Modeling of transpiration reduction in van Genuchten–Mualem type soils

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[1] We derive an analytic expression for the matric flux potential (M) for van Genuchten–Mualem (VGM) type soils which can also be written in terms of a converging infinite series. Considering the first four terms of this series, the accuracy of the approximation was verified by comparing it to values of M estimated by numerical finite difference integration. Using values of the parameters for three soils from different texture classes, the proposed four-term approximation showed an almost perfect match with the numerical solution, except for effective saturations higher than 0.9. Including more terms reduced the discrepancy but also increased the complexity of the equation. The four-term equation can be used for most applications. Cases with special interest in nearly saturated soils should include more terms from the infinite series. A transpiration reduction function for use with the VGM equations is derived by combining the derived expression for M with a root water extraction model. The shape of the resulting reduction function and its dependency on the derivative of the soil hydraulic diffusivity D with respect to the soil water content θ is discussed. Positive and negative values of dD/dθ yield concave and convex or S-shaped reduction functions, respectively. On the basis of three data sets, the hydraulic properties of virtually all soils yield concave reduction curves. Such curves based solely on soil hydraulic properties do not account for the complex interactions between shoot growth, root growth, and water availability.

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1. Introduction

[2] Soil water uptake by plants is controlled by water potential gradients and resistances in the soil-plant-atmosphere pathway. Soil water content, pressure head and hydraulic conductivity are interdependent soil properties. Models describing root water extraction must deal with varying pressure head gradients and hydraulic conductivities between the bulk soil and the root surface. In this context, the matric flux potential (or Kirchhoff transform), an integration of hydraulic conductivity K over a range of pressure heads h, was applied numerically and analytically by *de Jong van Lier et al.* [2006] and by *Metselaar and de Jong van Lier* [2007].

[3] For a transpiring plant extracting water from the soil, the actual soil water availability can be characterized as nonlimiting or limiting. As long as the soil water content is in the nonlimiting range or constant rate phase [*de Jong Van Lier et al.*, 2006], transpiration occurs at the potential rate with relative transpiration being constant and equal to 1. When soil water content falls below a certain threshold value (θ_l, m^3 m^{-3}), relative transpiration starts to decrease, reaching a minimum (zero) value at the permanent wilting point θ_w. Therefore, this limiting water range ($\theta_w \leq \theta < \theta_l$) is also called the falling rate phase [e.g., *Palmer et al.*, 1964; *Li et al.*, 2001; *Feddes and Raats*, 2004; *Kozak et al.*, 2005].

[4] Establishing the shape of the transpiration function by modeling the decay of relative transpiration as a function of decreasing soil water content is an important aspect in hydrological and meteorological models. Piecewise linear relations with a spatial average of soil water content [*Thornthwaite and Mather*, 1955; *Doorenbos and Kassam*, 1986] or pressure head [*Feddes et al.*, 1988] have been proposed. *Molz* [1981] reviewed macroscopic approaches to estimate transpiration rates under limiting hydraulic conditions. Nonlinear empirical relations have also been proposed [e.g., *Minhas et al.*, 1974; *van Genuchten*, 1987; *Skaggs et al.*, 2006]. In contrast to the above empirical functions, *Philip* [1957b], *Gardner* [1960], *Cowan* [1965] and others have analyzed root water uptake in terms of radial flow toward a single root. Development notably of analytical solutions in this area has been reviewed by *Raats* [2007].

[5] Using soils characterized by physical properties that yield flow equations that can be solved analytically, so-called analytical soils, in an analysis of radial water flow toward a single root, *Metselaar and de Jong van Lier* [2007] showed that the linear reduction function as proposed by *Thornthwaite and Mather* [1955] and *Doorenbos and Kassam* [1986] corresponds to a soil in which diffusivity is constant with water content. For four classes of soils,

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these authors showed the reduction of relative transpiration with water content to have a concave shape, whereas their numerical simulations for the van Genuchten–Mualem class of soils, hereafter referred to as the VGM class of soils, show reduction functions that may be partially convex.

[6] Soil databases frequently contain VGM parameters for the hydrological characterization of unsaturated soils [e.g., *Wösten et al.*, 1999; *Schaap and Leij*, 2000; *Wösten et al.*, 2001]. Although the VGM model usually matches experimental data satisfactorily, its functional form limits the availability of solutions for infiltration and drainage problems [*Ross*, 1992]. For very dry soils modifications have been introduced to describe water vapor adsorption [*Ross et al.*, 1991; *Rossi and Nimmo*, 1994]. Other models, like those of *Brooks and Corey* [1964], yield equations that are mathematically easier to manipulate.

[7] An important mathematical transformation, the matric flux potential M or Kirchhoff transform, is the integration of hydraulic conductivity K over a range of pressure heads h, and is frequently used in studies involving water flow in unsaturated nonswelling, nonhysteretic soils. It is especially useful in the modeling of flow phenomena where the spatial derivative of pressure head is the main driving force, e.g., the uptake of soil water by plant roots. *Cowan* [1965] pioneered the use of matric flux potential to analyze radial flow toward a root. Many years later, *Heinen* [2001] used an analytical solution based on the matric flux potential for radial flow toward a single root in an iterative procedure (FUSSIM2) to estimate actual transpiration. The matric flux potential also played an important role in the doctoral thesis and papers by *De Willigen and Van Noordwijk* [1987, 1991, 1994].

[8] The concept of matric flux potential has been used in experimental and theoretical analyses of flow problems on the basis of the assumption of an exponential dependence of the hydraulic conductivity K upon the pressure head h, the so-called quasi-linearization [*Pullan*, 1990]. *Ross and Bristow* [1990] and *Ross* [1992] have also used the matric flux potential and its approximation with success in numerical models for layered soils. Other applications of matric flux potential found in the literature include: simulation of rice production hydrology [*Ten Berge et al.*, 1995]; analysis of the Guelph permeameter data [*Elrick et al.*, 1995]; modeling infiltration [*Raats*, 1970; *Ragab et al.*, 1984; *Philip and Knight*, 1997]; inferring soil hydraulic parameters by analysis of steady state weight loss [*Ten Berge et al.*, 1987]; modeling soil evaporation [*Shaykewich and Stroosnijder*, 1977].

[9] Despite the physical importance of the matric flux potential and the frequent use of the VGM model, no analytical solution is available for the M-h or M-θ relations in a VGM type soil [*Tartakovsky et al.*, 2003]. For these soils, modelers in soil hydrology need to use numerical solutions [e.g., *de Jong van Lier et al.*, 2006], or must convert available VGM parameters to the Brooks and Corey equation which results in a loss of precision [*Morel-Seytoux et al.*, 1996; *Leij et al.*, 2005; *Haverkamp et al.*, 2005]. Hence, an analytical solution for M in terms of the VGM equation system would be a beneficial contribution. In this paper we analytically derive an expression for the matric flux potential for VGM type soils, describe its approximation using a converging infinite series and discuss its validity. Using the approach applied by *Metselaar and de Jong van Lier* [2007] and the newly derived expression for M, we discuss the shape of resulting reduction functions for van Genuchten–Mualem type soils.

2. Material and Methods

[10] Defining the matric flux potential M (m^2 d^{-1}) as the integral of hydraulic conductivity ($K(h)$, m d^{-1}) over pressure head h (m^{-1}), equivalent to the integral of diffusivity ($D(\theta)$, m^2 d^{-1}) over water content (θ, m^3 m^{-3}), and choosing the permanent wilting point in terms of pressure head (h_w, m) or water content (θ_w, m^3 m^{-3}) as the lower bound of the integral, we have

$$M = \int_{h_w}^{h} K(h)dh = \int_{\theta_w}^{\theta} D(\theta)d\theta \qquad (1)$$

[11] Assuming a homogeneous root distribution and uniform macroscopic soil water content as previously assumed by *Philip* [1957b], *Gardner* [1960] and *Cowan* [1965], *Metselaar and de Jong van Lier* [2007] showed that the relative transpiration T_r is equal to the relative mean matric flux potential:

$$T_r = \frac{T_a}{T_p} = \frac{M}{M_l} \qquad (2)$$

where T_a and T_P are the potential and the actual transpiration rate (m d^{-1}), M is the mean matric flux potential and M_l is the mean matric flux potential at the onset of the falling rate phase (corresponding to $\theta = \theta_l$).

[12] On the basis of equation (2), *Metselaar and de Jong van Lier* [2007] derived reduction curves for several classes of soils. However, the most frequently used hydraulic functions, combining the model for the water retention characteristic *of van Genuchten* [1980] with the model for prediction of the hydraulic conductivity characteristic of *Mualem* [1976], were not included in their analysis as the evaluation of the integral for M as defined by equation (1) was not available. The van Genuchten–Mualem equations are defined as [*Wösten and van Genuchten*, 1988]

$$\Theta = [1 + (-\alpha h)^n]^{-m} \qquad (3)$$

$$K(\Theta) = K_s \Theta^l \left\{1 - \left(1 - \Theta^{\frac{1}{m}}\right)^m\right\}^2 \qquad (4)$$

in which

$$\Theta = \frac{\theta - \theta_r}{\theta_s - \theta_r} \qquad (5)$$

with θ, θ_{r} and θ_{s} are the water content, the residual water content and the water content at saturation (m^3 m^{-3}), respectively, h ($h \le 0$) is the pressure head (m), K and K_s are the hydraulic conductivity and the hydraulic conductivity at saturation, respectively (m d^{-1}), and α (m^{-1}), m, n ($n > 1$), and l are empirical parameters, with $\alpha > 0$ and $0 < m < 1$.

Table 1. Soil Physical Parameters for Three Soils From the Dutch Staring Series[a]

Staring Soil Identification	Textural Class	Short Name in This Paper	van Genuchten Equation System					
			θ_r (m^3 m^{-3})	θ_s (m^3 m^{-3})	α (m^{-1})	l	n	K_s (m d^{-1})
B3	loamy sand	sand	0.02	0.46	1.44	−0.215	1.534	0.1542
B11	heavy clay	clay	0.01	0.59	1.95	−5.901	1.109	0.0453
B13	sandy loam	loam	0.01	0.42	0.84	−1.497	1.441	0.1298

[a]*Wösten et al.* [2001].

Van Genuchten [1980] showed that the hydraulic conductivity (equation (4)) is given by the equation derived by *Mualem* [1976] if $m = 1 - 1/n$ and $n > 1$, or by the Burdine theory [*Burdine*, 1953] if $m = 1-2/n$ and $n > 2$.

[13] In Appendix A, a closed expression is presented in terms of a hypergeometric function for $M(\Theta)$ that allows an approximation of equation (2) for the van Genuchten–Mualem class of soils. According to its fourth-order approximation:

$$T_r = \frac{M}{M_l} = \frac{\Lambda(\Theta) - \Lambda(\Theta_w)}{\Lambda(\Theta_l) - \Lambda(\Theta_w)} \qquad (6)$$

with

$$\Lambda(\Theta) = 2m\Theta^{a_1} + [(1+m)B_1 - (1-m)B_3]\Theta^{a_2} + [(1+m)B_1B_2 - (1-m)B_3B_4]\Theta^{a_3} \qquad (7)$$

and

$$\begin{aligned} a_1 &= \frac{1}{m} + l + 1 \quad B_1 = \frac{(1+\varphi)(2+m)}{3(2+\varphi)} \\ a_2 &= \frac{2}{m} + l + 1 \quad B_2 = \frac{(2+\varphi)(3+m)}{4(3+\varphi)} \\ a_3 &= \frac{3}{m} + l + 1 \quad B_3 = \frac{(1+\varphi)(2-m)}{3(2+\varphi)} \\ \varphi &= m(l+1) \quad B_4 = \frac{(2+\varphi)(3-m)}{4(3+\varphi)} \end{aligned} \qquad (8)$$

[14] The second derivative $d^2T_r/d\Theta^2$ is used to evaluate the shape of the function $T_r(\Theta)$. It follows from equation (2) that

$$\frac{d^2T_r}{d\Theta^2} = \frac{1}{M_l}\frac{d^2M}{d\Theta^2} \qquad (9)$$

Also, by definition, it follows that

$$D = \frac{dM}{d\theta} = \frac{1}{(\theta_s - \theta_r)}\frac{dM}{d\Theta} \Rightarrow \frac{d^2M}{d\Theta^2} = (\theta_s - \theta_r)\frac{dD}{d\Theta} \qquad (10)$$

Combining equations (9) and (10) we obtain

$$\frac{d^2T_r}{d\Theta^2} = \frac{(\theta_s - \theta_r)}{M_l}\frac{dD}{d\Theta} \qquad (11)$$

[15] According to equation (11), the sign of $dD/d\Theta$ determines the shape of the reduction curve. If $dD/d\Theta > 0$, the reduction curve will be concave. In the case that $dD/d\Theta$ equals zero over the full range of Θ (a constant diffusivity soil), the reduction curve will be linear. The case $dD/d\Theta < 0$ will result in a convex reduction curve.

3. Results and Discussion

3.1. Evaluation of the Solution Approximation for $M(\theta)$ With van Genuchten–Mualem Hydraulic Functions

[16] Using four terms of a converging infinite series, equation (A17) is an approximation of the Gauss hypergeometric function. Values of $M(\Theta)$ obtained from this equation were compared to those estimated by a numerical finite difference integration using a geometric average for K:

$$M(h_p) = \sum_{i=1}^{p} (h_i - h_{i-1})\sqrt{K(h_i)K(h_{i-1})} \qquad (12)$$

[17] In close correspondence to equation (1), h_0 in equation (12) equals h_M and was chosen, in the test cases, equal to −150 m, the common "permanent wilting point." The difference $(h - h_{i-1})$ was chosen in such a way that $\log(-h_i) - \log(-h_{i-1})$ was constant and equal to 0.001. Reducing this value did not significantly alter the results. The outcome of this analysis as a function of relative saturation Θ for the three soils from Table 1, illustrated in Figure 1, shows a significant divergence between both solutions starting approximately at $\Theta = 0.9$. As shown in Figure 2 for the clay soil, increasing the number of evaluated terms reduces this divergence, but very close to saturation convergence is slow. Slow convergence is a well known feature of the hypergeometric function for certain combinations of parameters. A greater number of evaluated terms increases the number of terms in equation (A19) required to evaluate the matric flux potential and may be justified for studies with special interest in water contents close to saturation. However, for studies not involving water contents close to water saturation, the observed divergence can be ignored. Therefore, we assumed that the analytical solution presented as equation (A17) is sufficiently accurate for the purpose considered here, transpiration reduction.

3.2. Reduction Curve for Soils With van Genuchten–Mualem Hydraulic Functions

[18] The physical implications of convex versus concave reduction functions are illustrated using a simple example.

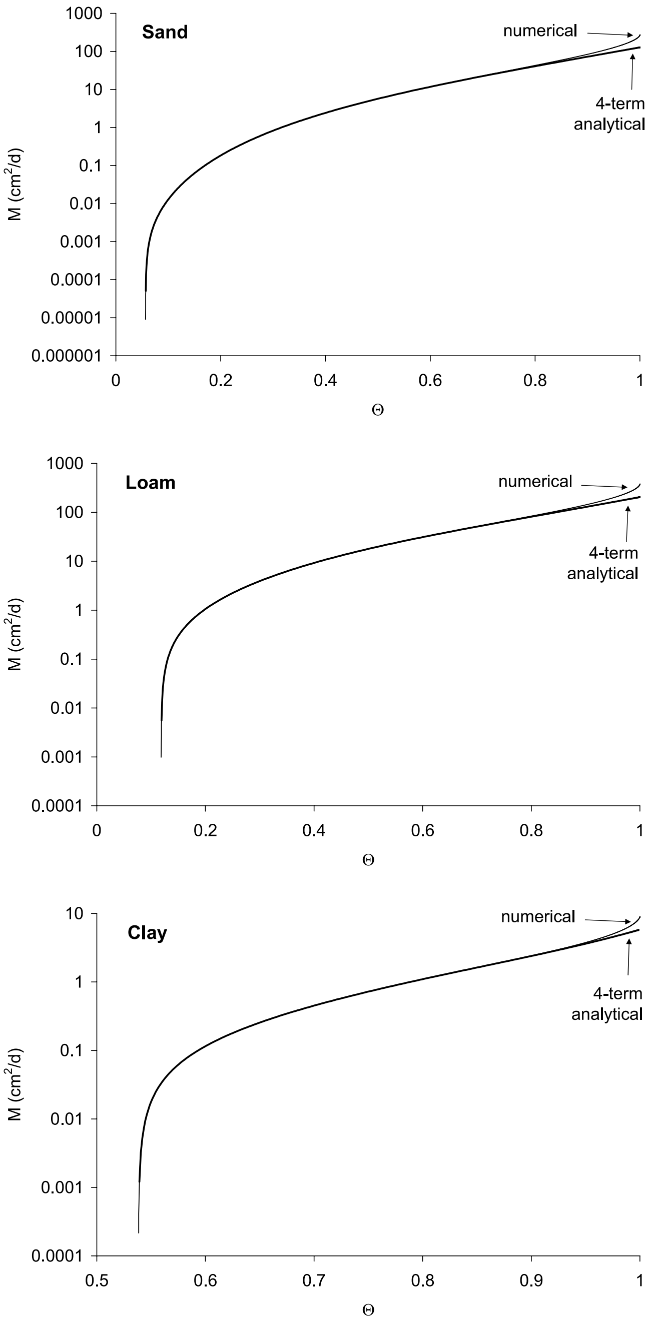

Figure 1. Matric flux potential M as a function of relative saturation Θ for sand, loam, and clay soil, according to the series expression with $k = 4$ (equation (A17)) and the numerical evaluation (equation (12)).

Assuming a transpiration rate T_a which is proportional to the water storage in the root zone (W, m), we have

$$T_a = -\frac{dW}{dt} = \left(\frac{W}{W_0}\right)^b T_p \tag{13}$$

where W_0 (m) is the storage at which transpiration is at its potential value, and b is a shape parameter. Rearranging equation (13), and assuming $W \leq W_0$, the corresponding reduction function is

$$\frac{T_a}{T_p} = \left(\frac{W}{W_0}\right)^b \tag{14}$$

which is concave over the entire range for $b > 1$, and convex for $0 < b < 1$. For $W > W_0$ the reduction function should be set equal to 1.

[19] The solution of equation (13), subject to $b \neq 1$ and $W = W_0$ at $t = t_0$, is

$$\frac{W}{W_0} = \left[1 - (1-b)\left(\frac{T_p}{W_0}(t - t_0)\right)\right]^{\frac{1}{1-b}} \tag{15}$$

[20] Defining a dimensionless time t' as

$$t' = \frac{T_p}{W_0} t \tag{16}$$

Equation (15) can be written as

$$W = W_0\left[1 - (1-b)\left(t' - t'_0\right)\right]^{\frac{1}{1-b}} \tag{17}$$

[21] Equation (17) shows that for $0 < b < 1$, i.e., for a convex reduction function, the storage W becomes zero at

$$t' = t'_0 + \frac{1}{1-b} \tag{18}$$

Thus, the physical consequence of a convex reduction function over the entire range of available moisture is a finite time for transpiration to become zero, and a finite range of time over which the function can be usefully applied.

[22] On the other hand for $b > 1$, i.e., a concave reduction function, equation (17) for $W = 0$ yields values of $t < t'_0$, which are physically meaningless. A concave function approaches zero transpiration asymptotically.

[23] Constraints on parameters for the van Genuchten–Mualem equations were discussed by *Fuentes et al.* [1991, 1992], *Durner et al.* [1999], and *Ippisch et al.* [2006]. *Durner et al.* [1999] showed that, in order to avoid physically unexpected behavior with $dK(\Theta)/d\Theta < 0$, the range of values permitted for l is restricted to

$$l > \frac{-2\Theta^{\frac{1}{m}}\left(1 - \Theta^{\frac{1}{m}}\right)^{m-1}}{1 - \left(1 - \Theta^{\frac{1}{m}}\right)^{m}} \tag{19}$$

which for $\Theta \to 0$ can be shown to be equivalent to

$$l > \frac{-2}{m} \tag{20}$$

The condition from equation (20) warrants that $dK/d\Theta > 0$ for any Θ. For $l \geq -2$ and $0 < m < 1$, any value of m yields a physically consistent behavior. For $l < -2$, equation (20) must be satisfied to find the valid range of m.

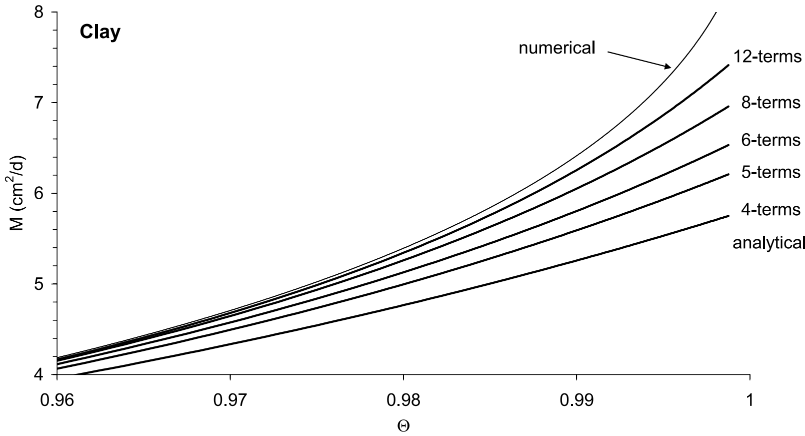

Figure 2. Matric flux potential M as a function of relative saturation Θ for clay soil in the nearly saturated zone, according to the series expression with k = 4, 5, 6, 8, and 12, and according to the numerical evaluation (equation (12)).

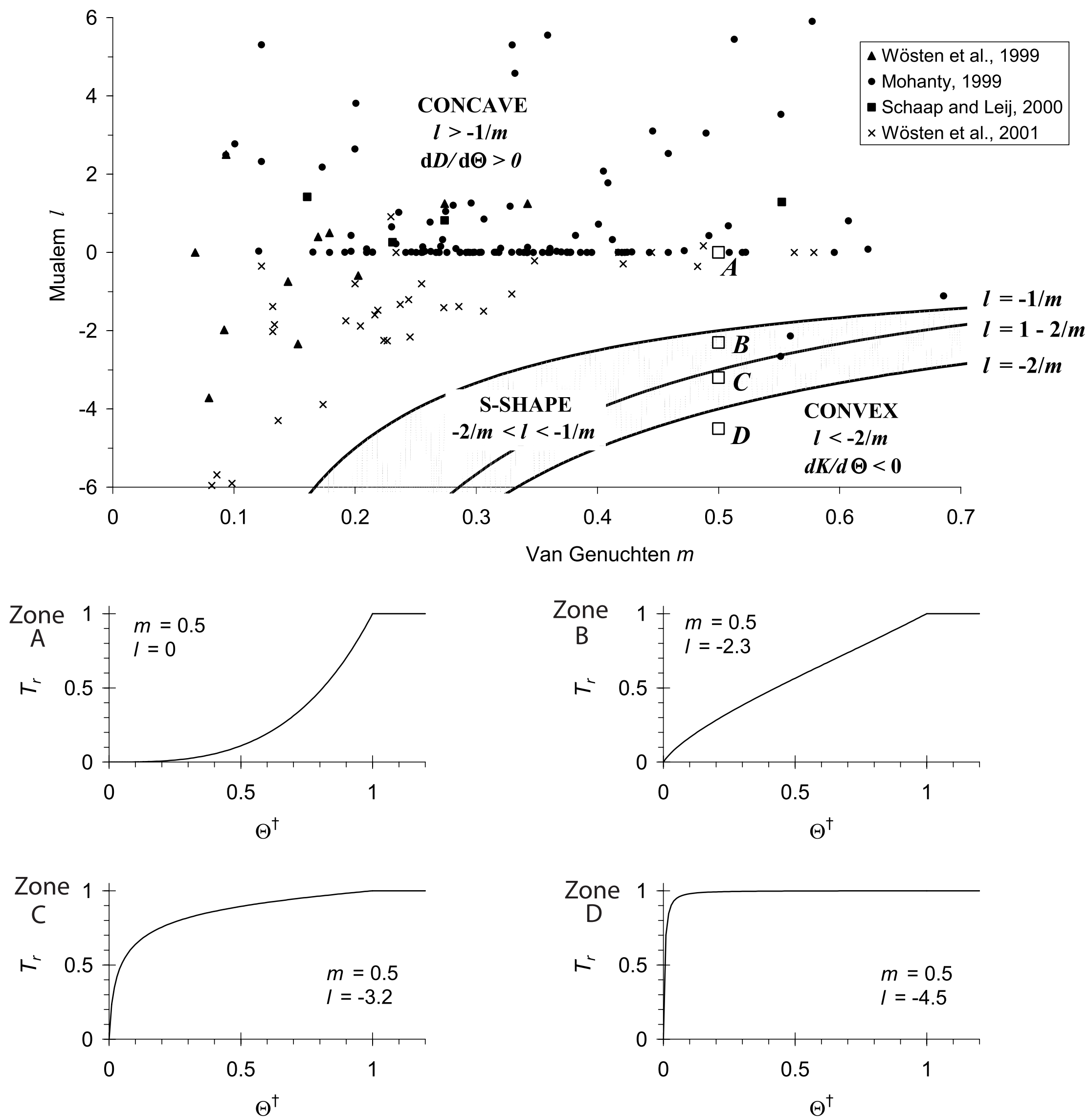

Figure 3. (top) Reduction function shape zones for combinations of van Genuchten parameter m and Mualem parameter l, together with soils from *Wösten et al.* [1999], *Schaap and Leij* [2000], and *Wösten et al.* [2001] and the Mohanty (1999) database. (bottom) Reduction curves for four combinations of parameters m and l indicated in the top plot; $\Theta^\dagger = (\theta - \theta_w)/(\theta_l - \theta_w)$.

[24] It can also be shown that the condition

$$l > \frac{-1}{m} \quad (21)$$

results in $dD/d\Theta > 0$ for any Θ. In the range $-2/m \leq l \leq -1/m$, $dD/d\Theta$ is positive for smaller values of Θ and negative for greater values.

[25] Furthermore, when the Poiseuille capillary flow equation is considered for hydraulic equilibrium conditions, the second derivative $d^2K/d\Theta^2$ cannot be negative. *Fuentes et al.* [1992] found the combination of the van Genuchten equation (equation (3)), the Burdine theory with $m = 1 - 2/n$ [*van Genuchten*, 1980] and the Brooks and Corey hydraulic conductivity equation to be the least constraining when using the *Philip* [1957a] infiltration equation applied over a relatively long period of time. From their analysis it can be shown that the condition $d^2K/d\Theta^2 > 0$ imposes the following constraint on the van Genuchten parameters:

$$l > 1 - \frac{2}{m}. \quad (22)$$

[26] The impact of equations (20), (21), and (22) on the shape of the reduction function is illustrated in Figure 3 where the Mualem parameter in the range $-6 \leq l \leq 6$ is plotted against the van Genuchten parameter m between 0 to 0.7. Four zones, designated as A through D, can be located from top to bottom in Figure 3. At the top, in zone A where $dD/d\Theta > 0$, $dK/d\Theta > 0$ and $d^2K/d\Theta^2 > 0$ for all values of Θ, the reduction function is concave. In zone B where $dD/d\Theta$ is positive in the range of higher water contents and negative in the range of lower water contents, $dK/d\Theta > 0$ and $d^2K/d\Theta^2 > 0$, the reduction function is S shaped. In zone C, $dD/d\Theta$ is positive in the range of higher water contents and negative in the range of lower water contents, $dK/d\Theta > 0$, but $d^2K/d\Theta^2 < 0$ which demands an improbable behavior of the hydraulic conductivity. And at the bottom, zone D where $dD/d\Theta < 0$, $dK/d\Theta < 0$ and $d^2K/d\Theta^2 < 0$ for all values of Θ is physically impossible. Examples of reduction curves, labeled A through D for each zone calculated from equation (6) with $m = 0.5$ and $l = 0$, -2.3, -3.2 and -4.5, respectively, are shown at the bottom of Figure 3. The coordinates (0.5, l) of the four examples are indicated by open squares in the top plot of Figure 3.

[27] The van Genuchten and Mualem parameter values (m, l) for soils from four well-documented databases were also plotted in Figure 3. The data from HYPRES [*Wösten et al.*, 1999] are average values for 11 textural and pedological classes, obtained for European soils. The Mohanty database (B. P. Mohanty, The Southern Great Plains 1997 (SGP97) soil property measurement data set, 1999, available at http://daac.gsfc.nasa.gov/fieldexp/SGP97/arssl.html#5052) contains data from 127 soils from the Southern Great Plains. *Schaap and Leij* [2000] soils refer to average hydraulic parameters for four textural groups (sands, loams, silts and clays) from the UNSODA database, mainly from the temperate zone in the northern hemisphere. *Wösten et al.* [2001] report average parameter values for 18 surface and 18 subsurface layers of soil classes from the Netherlands. With the exception of only two soils from the Mohanty database, a close examination of Figure 3 reveals that all of the above data are within the region of concave reduction curves. Those two soils reported by Mohanty are in the S-shaped reduction curve zone, with one of them manifesting the unexpected behavior of $d^2K/d\Theta^2 < 0$.

4. Conclusions

[28] An analytical expression for the matric flux potential M for the van Genuchten–Mualem class of soils, derived using a Gauss hypergeometric function transform, was written in terms of a converging infinite series. Except for a small range of water contents very close to saturation, the sum of the first four terms of this series did not diverge significantly from numerically calculated values of M. Hence, the truncated analytic expression can be used for applications at intermediate soil water contents. For infiltration problems or analysis of flow in nearly saturated soils, more terms from the infinite series should be included.

[29] For any soil, the shape of the transpiration reduction curve depends on the sign of $dD/d\theta$, with positive, zero, and negative values corresponding, respectively, to concavity, linearity, and convexity. Specifically for the van Genuchten–Mualem class of soils, an overall positive $dD/d\theta$, corresponding to $l > -1/m$, results in concave reduction functions. An overall negative $dD/d\theta$, for $l < -2/m$, corresponds to physically unexpected behavior. Between these two limits, when $-2/m \leq l \leq -1/m$, an S-shaped reduction curve results, convex for lower and concave for higher water contents. Within this zone, only soils with $l > 1-2/m$ meet the condition $d^2K/d\Theta^2 > 0$. With the exception of two soils manifesting S-shaped reduction curves, data sets for 180 soils show that the soil hydraulic properties result in concave reduction curves. This analysis of the shape of the reduction function is based on a soil physical point of view, and does not consider the complex interactions between shoot growth, root growth and water availability.

Appendix A: $M(\theta)$ for the van Genuchten–Mualem Class of Soils

A1. Derivation of the Expression for $M(\theta)$

[30] The soil hydraulic diffusivity (D, $m^2\ d^{-1}$) is defined as the relationship between the soil hydraulic conductivity (K, $m\ d^{-1}$) and the specific water content ($C = d\theta/dh$, m^{-1}):

$$D = \frac{K}{C} \quad (A1)$$

where using the van Genuchten–Mualem hydraulic functions:

$$C(\Theta) = \alpha(n-1)(\theta_s - \theta_r)\Theta^{\frac{1}{m}}\left(1 - \Theta^{\frac{1}{m}}\right)^m \quad (A2)$$

Combining the definition of the matric flux potential (equation (1) with equation (5)) and evaluating for a water content θ_a corresponding to Θ_a it follows that

$$M(\theta_a) = \int_{\theta_w}^{\theta_a} D(\theta)d\theta = (\theta_s - \theta_r)\int_{\Theta_w}^{\Theta_a} D(\Theta)d\Theta \quad (A3)$$

Substituting equations (4) and (A2) in (A1), and equation (A1) in (A3) yields

$$M(\Theta_a) = \frac{K_s}{\alpha(n-1)} \int_{\Theta_w}^{\Theta_a} \frac{\Theta^l \left\{1 - \left(1 - \Theta^{\frac{1}{m}}\right)^m\right\}^2}{\Theta^{1/m}\left(1 - \Theta^{\frac{1}{m}}\right)^m} d\Theta \tag{A4}$$

[31] To evaluate the integral in (A4), the following transformation is introduced:

$$x = \Theta^{\frac{1}{m}} \tag{A5}$$

then:

$$\Theta = x^m \tag{A6}$$

and

$$d\Theta = mx^{m-1}dx \tag{A7}$$

Substituting (A6) and (A7) in (A4) yields

$$M(\Theta_a) = \frac{mK_s}{\alpha(n-1)} \int_{\Theta_w^{1/m}}^{\Theta_a^{1/m}} \frac{x^{lm}\{1-(1-x)^m\}^2}{x(1-x)^m} x^{m-1}dx \tag{A8}$$

which can be rewritten as

$$M(\Theta_a) = \frac{mK_s}{\alpha(n-1)} \int_{\Theta_w^{1/m}}^{\Theta_a^{1/m}} x^{m(l+1)-2}[(1-x)^{-m} - 2 + (1-x)^m]dx \tag{A9}$$

[32] Equation (A9) is a well defined incomplete Beta function [*Abramowitz and Stegun*, 1972, p. 263, equation 6.6.1], and can be expressed in relation to the Gauss hypergeometric function as [*Abramowitz and Stegun*, 1972, p. 263, equation 6.6.8]

$$M(\Theta_a) = \frac{mK_s}{\alpha(n-1)[\varphi-1]}(f_1 + f_2 - 2)\left(\Theta_a^{l+1-\frac{1}{m}} - \Theta_w^{l+1-\frac{1}{m}}\right) \tag{A10}$$

or, considering $m = 1-1/n$:

$$M(\Theta_a) = \frac{K_s}{\alpha n[\varphi-1]}(f_1 + f_2 - 2)\left(\Theta_a^{l+1-\frac{1}{m}} - \Theta_w^{l+1-\frac{1}{m}}\right) \tag{A11}$$

where

$$\varphi = m(l+1) \tag{A12}$$

and with

$$f_1 = {}_2F_1\left[\varphi - 1, m; \varphi; \Theta^{\frac{1}{m}}\right] \tag{A13}$$

and

$$f_2 = {}_2F_1\left[\varphi - 1, -m; \varphi; \Theta^{\frac{1}{m}}\right] \tag{A14}$$

where ${}_2F_1$ represents the Gauss hypergeometric function [*Abramowitz and Stegun*, 1972].

[33] Equations (A10) and (A11) represent exact expressions for the matric flux potential for van Genuchten–Mualem type soils. Both expressions are undefined for $\varphi = 1$, equivalent to $l = (1/m) - 1$. For the case of root water uptake we explicitly assume Θ_w not to become 0. If, however, this condition is required for an analysis the equations only yield real numbers for the matric flux potential for $l > (1/m) - 1$.

A2. Infinite Series Approximation for *M*(*θ*) With van Genuchten–Mualem Hydraulic Functions

[34] To evaluate the hypergeometric function (${}_2F_1$) embedded in equation (A10), it can be written as an infinite converging series [cf. *Abramowitz and Stegun*, 1972, p. 556] according to

$$\begin{aligned} f_1 &= {}_2F_1\left[\varphi - 1, m; \varphi; \Theta^{\frac{1}{m}}\right] \\ &= \sum_{k=0}^{\infty}\left\{\frac{[\varphi-1]_k (m)_k}{[\varphi]_k}\cdot\frac{\Theta^{\frac{k}{m}}}{k}\right\} \\ &= 1 + \sum_{k=1}^{\infty}\left[\frac{\Theta^{\frac{k}{m}}}{k!}\prod_{j=1}^{k}\frac{[\varphi+j-2][m+j-1]}{[\varphi+j-1]}\right] \end{aligned} \tag{A15}$$

where $[\varphi - 1]_k$, $(m)_k$, and $[\varphi]_k$ are Pochhammer symbols. Analogously,

$$\begin{aligned} f_2 &= {}_2F_1\left[\varphi - 1, -m; \varphi; \Theta^{\frac{1}{m}}\right] \\ &= 1 + \sum_{k=1}^{\infty}\left[\frac{\Theta^{\frac{k}{m}}}{k!}\prod_{j=1}^{k}\frac{[\varphi+j-2][j-m-1]}{[\varphi+j-1]}\right] \end{aligned} \tag{A16}$$

[35] Evaluating equation (A15) for the first four terms (i.e., $k = 1, 2, 3$, and 4), we obtain a fourth-order approximation for M:

$$M(\Theta_a) = \frac{m(1-m)K_s}{2\alpha[\varphi+1]}[\Lambda(\Theta_a) - \Lambda(\Theta_w)] \tag{A17}$$

with

$$\Lambda(\Theta) = \Theta^{\frac{1+\varphi}{m}}\left\{\begin{array}{l}(1+m)\left\{1 + \frac{\Theta^{\frac{1}{m}}}{3}\frac{(1+\varphi)(2+m)}{(2+\varphi)}\left[1 + \frac{\Theta^{\frac{1}{m}}}{4}\frac{(2+\varphi)(3+m)}{(3+\varphi)}\right]\right\} - \\ (1-m)\left\{1 + \frac{\Theta^{\frac{1}{m}}}{3}\frac{(1+\varphi)(2-m)}{(2+\varphi)}\left[1 + \frac{\Theta^{\frac{1}{m}}}{4}\frac{(2+\varphi)(3-m)}{(3+\varphi)}\right]\right\}\end{array}\right\} \tag{A18}$$

Equation (A18) can also be written as

$$\Lambda(\Theta) = 2m\Theta^{a_1} + [(1+m)B_1 - (1-m)B_3]\Theta^{a_2} + [(1+m)B_1B_2 - (1-m)B_3B_4]\Theta^{a_3} \quad \text{(A19)}$$

where

$$\begin{aligned} a_1 &= \frac{1}{m} + l + 1 \quad & B_1 &= \frac{(1+\varphi)(2+m)}{3(2+\varphi)} \\ a_2 &= \frac{2}{m} + l + 1 \quad & B_2 &= \frac{(2+\varphi)(3+m)}{4(3+\varphi)} \\ a_3 &= \frac{3}{m} + l + 1 \quad & B_3 &= \frac{(1+\varphi)(2-m)}{3(2+\varphi)} \\ & & B_4 &= \frac{(2+\varphi)(3-m)}{4(3+\varphi)} \end{aligned} \quad \text{(A20)}$$

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